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Gravitation

Newton's universal gravitational law.

The law states that *every particle of matter in the universe attracts every other particle with a force which is directly proportional to the productof their masses and inversely proportional to the square of the distance between them.*

Consider two bodies of masses $m1$ and $m2$ with their centres separated by a distance r. The gravitational force between them is

$$F \propto m_1 m_2 \qquad and \qquad F \propto \frac{1}{r^2} \qquad \therefore F = \frac{G m_1 m_2}{r^2}$$

where G is the universal gravitational constant.

If $m_1 = m_2 = 1$ kg and $r = 1$ m, then $F = G$.

Hence, the Gravitational constant 'G' is numerically equal to the gravitational force of attraction between two bodies of mass 1 kg each separated by a distance of 1 m.

The value of G is 6.67×10^{-11} N m^2 kg^{-2}

and its dimensional formula is $M^{-1} L^3 T^{-2}$.

Acceleration due to gravity

Galileo found that *"in the absence of air, all bodies will fall at the same rate"*.

Experiments showed that the velocity of a freely falling body under gravity increases at a constant rate. (i.e) with a constant acceleration.

The acceleration produced in a body on account of the force of gravity is called *acceleration due to gravity.*

It is denoted by g..

Value of g at sea–level and at a latitude of 45^o is taken as the standard (i.e) $g = 9.8$ m s^{-2}

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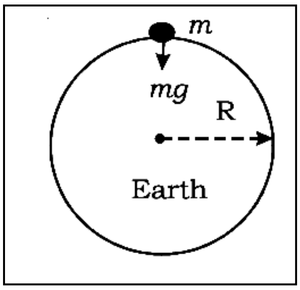
Acceleration due to gravity at the surface of the Earth



Let us consider a body of mass m on the surface of the Earth . Its distance from the centre of the Earth is R (radius of the Earth).

The gravitational force experienced by the body is

$F = \dfrac{GMm}{r^2}$, where M is the mass of the Earth.

From Newton's second law of motion, Force $F = mg$.

Equating the above two forces, $F = \dfrac{GMm}{r^2}$ = mg => $g = \dfrac{GM}{r^2}$

This equation shows that g is independent of the mass of the body m.

Mass of the Earth

$$M = \frac{gR^2}{G} = \frac{9.8\times(6.38\times10^6)^2}{6.67\times10^{-11}} = 5.98 \times 10^{24}\ \text{kg}$$

Variation of acceleration due to gravity

(i) Variation of g with height or altitude

Let P be a point on the surface of the Earth and Q be a point at an altitude h. Let the mass of the Earth be M and radius of the Earth be R.

The acceleration due to gravity at P on the surface is $g = \dfrac{GM}{R^2}$ ……………... (1)

Let the body be placed at Q at a height h from the surface of the Earth. The acceleration due to gravity at Q is $g_h = \dfrac{GM}{(R+h)^2}$ …………..... (2)

dividing (2) by (1), $\frac{g_h}{g} = \frac{R^2}{(R+h)^2} = \frac{R^2}{R\left(1+\frac{h}{R}\right)^2} = \left(1+\frac{h}{R}\right)^{-2}$ ....(3)

By using binomial theorem ,

$$(1+x)^n = 1+nx, if \quad x \quad << 1$$

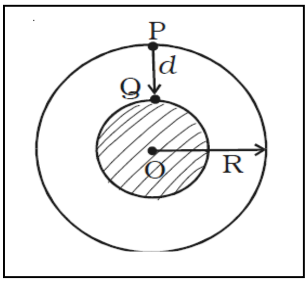
$$As \ \frac{h}{R} << 1 \quad => \left(1+\frac{h}{R}\right)^{-2} = \left(1-\frac{2h}{R}\right).......(3)$$

Putting (4) in (3)

$$(3) => g_h = g\left(1-\frac{2h}{R}\right)............................(4)$$

The value of acceleration due to gravity decreases with increase in height above the surface of the Earth.

(ii) Variation of g with depth



Let us consider the Earth to be a homogeneous sphere with uniform density of radius *R* and mass *M*.

Let *P* be a point on the surface of the Earth and *Q* be a point at a depth *d* from the surface. The acceleration due to gravity at *P* on the surface is

$$g = \frac{GM}{R^2} \ ....(1)$$

∴ If ρ be the density, then, the mass of the Earth is

$$M = \frac{4}{3}\pi R^3 \rho \ ....(2)$$

So acceleration due to gravity in terms density ρ

$$g = \frac{4}{3}\pi GR\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The acceleration due to gravity at Q at a depth d from the surface of the Earth is

$$g_d = \frac{4}{3}\pi G(R-d)\rho$$

dividing (2) by (1),

$$\Rightarrow \frac{g_d}{g} = \frac{(R-d)}{R} \quad \Rightarrow g_d = \left(1-\frac{d}{R}\right)g$$

The value of acceleration due to gravity decreases with increase of depth.

Q. Differentiate Inertial mass & gravitational mass.-

The mass measured with the help of Newton's 2nd law is known as Inertial mass. While the mass measured from the Newtons law of gravitation attraction is known as gravitational mass.

Inertial mass is the measurement of inertia of a body. While gravitational mass is the measurement of gravitation pull of the body.

Inertial mass, m = F/a

Gravitational mass mg = Fr^2/GM.

Q. What is Gravitational field:-

The field created around a material body in which it can experience Newton's law of gravitational attraction is known as gravitation Field.

Q. What is Gravitational Intensity? Find an expression for it .

Gravitational Intensity at a point in a gravitational field is defined as the force of gravitational attraction experienced by a unit mass when placed at that point.

I = F/m …………………(i)

Let P be a point at a distance of r from the centre (c) of the earth of mass (M)

Force of gravitation , $F = GM\, m / r^2$………. (ii)

So gravitational intensity at P

$I = GM/r^2$ ……………… (iii) Unit of I = N/kg

On the surface earth gravitation into is equal to the accelerant due to gravity.

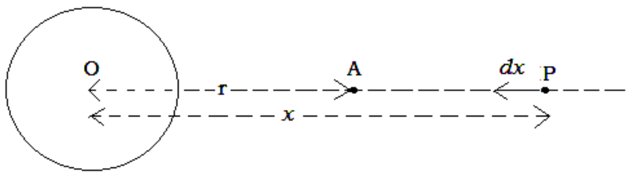
We know $I = GM/r^2$

On the surface of earth r = R(radius of earth) ,

$\mathbf{I = GM/R^2 = g}$

Q. What is Gravitational Potential? Find an expeession for it

Gravitational Potential at a point in a gravitational field is different as the amount of work done in bringing a unit mass from infinity to that point.



Let A be a point at a distance r from centre of the earth , at which the gravitational potential is.

$V = W / m$………………..(i)

Let P be a point at a distant of x from the contra of the earth such that. $r < x < \alpha$ .From force of gravitation attraction at P

$E = GMm/x^2$ -------------- (ii)

Let Q be a point very much near to P such that PQ = dx. Amount of work done in brining it from P to Q , $dW = F\,dx$ …………………(iii).

Total work done in brining it from infinity to its present position.

$$W = \int_0^w dw = \int_x^r Fdx$$

$$\Rightarrow W = \int_\alpha^r \frac{GMm}{x^2} dx = GMm \int_\alpha^r dx\ x^{-2} = GMm \left[-\frac{1}{x}\right]_\alpha^r$$

$$\Rightarrow \frac{W}{m} = GM \left[\frac{1}{r} - \frac{1}{\alpha}\right]$$

$$\therefore v = -\frac{GM}{r} \text{.........} .(iii)$$

Q. What is gravitational potential energy ? Find out an expression for it ?

Ans : -Gravitational Potential energy of a satellite at a certain position is the amount of work done in being brining if from infinite to its position.

Let P be a point at a distance x from the centre of earth such that $r < x < \alpha$ at which gravitational force,

$F = GMm / r^2$ …….. (i)

Where G is universal gravitational force and R is Radius of the earth.

Let Q be another point very much closer to (P) such that PQ = dx, amount of work done in bringing it from P to Q is dW = Fdx ------- (iii)

Total work done in brining it from infinity to A,

$$W = \int_0^w dw = \int_x^r Fdx = \int_\alpha^r \frac{GMm}{k^2} dx$$

$$= GMm \int_\alpha^r x^{-2} dx = GMm \left[-\frac{1}{x}\right]_\alpha^r = -GMm \left[\frac{1}{r} - \frac{1}{\alpha}\right]$$

$$\therefore w = -\frac{GMm}{r} \text{.........} .(iv)$$

So potential energy in its orbit. **U = - GMm / r** ------------- (v)

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Q. Relation between Gravitational potential & gravitational intensity

Let 'A' be a point at a distant of r from the centre of earth of which gravitational potential & gravitational intensity.

$V = -GM / r$ . ... (i)

$I = -GM / r^2$ .... (ii)

(i) ÷ (ii) => $I = V / r$

In form of calculus, $I = dV/dr$ ---------- (ii)

O. State the Kepler's laws.

Ans. The three laws of Kepler can be stated as follows:

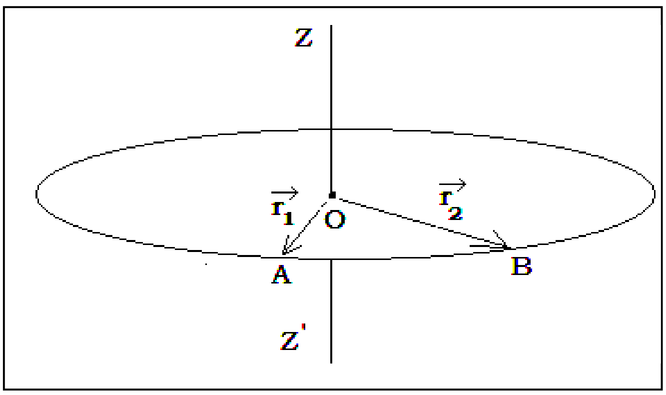
1. Law of orbits : All planets move in elliptical orbits with the Sun situated at one of the foci of the ellipse

2. Law of areas : The line that joins any planet to the sun sweeps equal areas in equal intervals of time.

3. Law of periods : The square of the time period of revolution of a planet is proportional to the cube of the semi-major axis of the ellipse traced out by the planet.

Q. Prove keplers 2nd Law

Let us consider a body of mass, m undergoing uniform circular motion along a circular path of radius,r with about the axis (zoz$^{/}$) .



Let with in a very short interval of time it moves from A to B with $\bar{r}_1$ & $\bar{r}_2$ as the position vector of A & B respect.

$|\overline{r_1}| = |\overline{r_2}| = |\overline{r}|$ ---------- (i)

From ΔAOB, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\Rightarrow \overrightarrow{AB} = \vec{r_2} - \vec{r_1} \Rightarrow \overrightarrow{AB} = d\vec{r}$

Area covered by the body in time (dt)

$$|d\vec{s}| = \frac{1}{2} \times |\overrightarrow{OA}| \times |\overrightarrow{AB}|$$

$$|d\vec{s}| = \frac{1}{2} \times |\vec{r}| \times |d\vec{r}| ----(iii)$$

So, areal velocity of the body,

$$\frac{|d\vec{s}|}{dt} = \frac{1}{2} \times |\vec{r}| \times \left|\frac{d\vec{r}}{dt}\right|$$

$$\frac{1}{2} \times |\vec{r}| \times |\vec{v}| = \frac{1}{2\text{m}} \times |\vec{r}| \times |v\overrightarrow{m}|$$

$$= \frac{1}{2m} \times |\vec{r}| \times |\vec{P}|$$

$$= \frac{1}{2m} \times |\vec{r} \times \vec{p}| \qquad \frac{|d\vec{s}|}{dt} = \frac{|\vec{L}|}{2m} ------(iv)$$

Where (L) is the angular momentum of the body.

From the ppl of conservation of angular momentum for an isolated system $\vec{L}$ is constant

$\frac{|d\vec{s}|}{dt} =$ constant ,i.c. areal velocity is constant.

This is the proof of Kepler's 2nd law.

Proof of Kepler's 3rd law from Newton's law of gravitation

Let (m) be the mass of a planet revolving round the sum of mass (M) in a circular orbit of radius (R) with velocity (V) & (T) as time period.

Force of gravitation between them

$F = GMm/R^2$ …. (i) , Where G is universal gravitational constant

The required centripetal force is given as

$F_c = mv^2/R$ ……………… (ii)

Again, $V = \frac{circumfere\ nce}{time\ \ period}$ => $v = 2\pi R/T$ …………… (iii)

$$\text{In equilibriu m,} \quad F_e = F_g$$

$$\Rightarrow \frac{mv^2}{R} = \frac{GMm}{R^2}$$

$$\Rightarrow v^2 = \frac{GM}{R}$$

$$\Rightarrow (\frac{2\pi R}{T})^2 = \frac{GM}{R}$$

$$\Rightarrow \frac{4\pi^2 R^2}{T^2} = \frac{GM}{R}$$

$$\Rightarrow T^2 = \frac{4\pi^2 R^3}{GM} \quad \therefore T^2 \ \alpha \ R^3$$

This is the Kepler's 3rd law from New ton's Law of gravitation.

Proof of Newton's law of gravitation from Kepler's 3rd law –

Let m be the mass of a plant revolving round the sun of mass M in a circular orbit of radius R with velocity v & T as time period.

The required centripetal force is given as

$F_c = mv^2/R$ …………. (i)

And, $v = (\frac{2\pi R}{T})$ ………. (ii)

From kepler's 3rd law, $T^2 \alpha R^3 \Rightarrow T^2 = KR^3$ ……….. (iii)

Putting (ii) in (i)

$$F_c = m\left(\frac{2\pi R}{T}\right)^2 / R = \frac{4\pi^2 Rm}{T^2}$$

$$\Rightarrow F_c = \frac{4\pi Rm}{KR^3} \quad \text{[using (iii)]}$$

$$\Rightarrow F = \frac{4\pi^2}{K}\frac{m}{R^2} ------(iv)$$

Assuming (1) As F is a force of mutual attraction between the planet & the sun. so it should be proportional to the mass sun.

2) Again we are measuring the force in the universe so presence of universal gravitational constant is also essential so replicing $4\pi^2/K$ by GM we have.

$$F = \frac{GMm}{R^2}$$

This is Newton's law of gravitation from kepler's 3rd law.

Escape velocity:-

The escape speed is the minimum speed with which a body must be projected in order that it may escape from the gravitational pull of the planet.

Consider a body of mass m placed on the Earth's surface. The gravitational potential energy is $E_P = -\frac{GMm}{R} - - - - - - (i)$

where M is the mass of the Earth and R is its radius. If the body is projected up with a speed v_e the kinetic energy is $K=\frac{1}{2}mv_e^2$ **……………..(2)**

∴ the initial total energy of the body is

$$E_i = E_P + K = -\frac{GMm}{R} + \frac{1}{2}mv_e^2 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

If the body reaches a height h above the Earth's surface, where the velocity be v , then the final total energy of the body at the height is

$$E_f = -\frac{GMm}{R+h} + \frac{1}{2}mv^2 \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad (4)$$

By the principle of conservation of mechanical energy

$$E_i = E_f \Rightarrow -\frac{GMm}{R} + \frac{1}{2}mv_e^2 = -\frac{GMm}{R} + \frac{1}{2}mv_e^2$$

The body will escape from the Earth's gravity at a height where the gravitational field ceases out , i.e. $h = \infty$.

At the height $h = \infty$, the speed v of the body is zero.

$$\text{Thus, } -\frac{GMm}{R} + \frac{1}{2}mv_e^2 = 0$$

$$\Rightarrow v_e = \sqrt{\frac{2GM}{R}} \ldots\ldots\ldots\ \ldots\ldots\ldots (5)$$

From the relation $g = \frac{GM}{R^2}$, we get $GM = gR^2$

Thus, the escape speed is $v_e = \sqrt{2gR} \ldots\ldots\ldots\ldots(6)$

The escape speed for Earth is 11.2 km/s, for the planet Mercury , it is 4 km/s and for Jupiter it is 60 km/s. The escape speed for the moon is about 2.5 km/s.